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Bose-like oscillator in fractional-dimensional space

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Abstract

The system of a fractional-dimensional Bose-like oscillator of one degree of freedom whose canonical variables satisfy the general Wigner commutation relations is investigated. Momentum–position uncertainty relations are obtained. For states without definite parity the well known one-dimensional Heisenberg uncertainty principle is retained, but for odd and even states momentum–position uncertainty inequalities depending on the dimension of the space are obtained. The motions of both the free particle and the harmonic oscillator in a fractional-dimensional space are studied through the probability density function. The existence of *compression* (*spread*) of the probability density for dimensions $D < 1$ ($D > 1$) is shown. Fractional-dimensional Bose-like operators are also deduced. They together with the reflection operator form an R-deformed Heisenberg algebra with a deformation parameter depending on the dimension of the space.

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1. Introduction

Studies on physical systems assuming continuous variation in the space dimension have been increasing over the last few decades. Critical phenomena [1–4] and fractal structures [5,6] have been extensively studied within the non-integer-dimensional space approaches. The possibility that the space–time dimension is slightly different from four, has also been considered by several authors [7–10].

Recently, a considerable amount of work has been devoted to the study of low-dimensional systems within the framework of the fractional-dimensional space model proposed by Stillinger [11]. By applying an approach introduced by He [12], in which the real semiconductor heterostructure system is substituted by an effective isotropic environment with a fractional dimension, exciton [13–15], magnetoexciton [16–18] and impurity [14, 19] states in semiconductor nanostructures have been successfully modelled. The dimensional parameter is then assumed as a measure of the degree of anisotropy or confinement of the real system. Thus, given this simple value, the real system can be modelled in a simple analytical way. However, the fractional-dimensional space is not a vector space [11], consequently, great difficulties arise in handling the basic formalism. Moreover, the difficulties present in both the geometrical and physical understanding of the formalism make the development of the fundamental properties and concepts of fractional-dimensional approaches in a systematic way difficult, dealing with the inclusion of a lot of *anzats* and *a priori* definitions (see for instance [13, 20, 21]).

One of the purposes of this paper is to find, in a systematic way, expressions for quantum mechanical operators that are fundamental for the description of systems of one degree of freedom in a fractional-dimensional space. The other one is to carry out a study of the behaviour of two simple systems (the free particle and the harmonic oscillator) in order to reveal the physical meaning of the dimensional parameter. This paper is organized as follows. For completeness of arguments we recall some results of previous works [22–26] in section 2, where an expression for the fractional-dimensional momentum operator corresponding to a system of one degree of freedom is presented. In section 3 momentum–position uncertainty relations in a fractional-dimensional space are obtained. The motion of a free particle in a fractional-dimensional space is investigated in section 4. In section 5 the behaviour of a fractional-dimensional Bose-like oscillator is carefully studied. The fractional-dimensional Bose-like operators are investigated in section 6, and conclusions are summarized in section 7.

2. Momentum operator

In order to study a system of a single degree of freedom in a fractional-dimensional space, we introduce a Cartesian-like *pseudocoordinate* ξ $(-\infty < \xi < \infty)$. Thus the radial integration weight (see [11]) may be written as

$$\sigma(D)r^{D-1} = \frac{\sigma(D)}{2}\,|\xi|^{D-1} \qquad \sigma(D) = \frac{2\pi^{D/2}}{\Gamma(D/2)} \tag{1}$$

where $\Gamma(x)$ is the gamma function.

In this way, the volume of the radius-R_0 sphere in the fractional-dimensional space is given by

$$V(R, D) = \int_0^{R_0} \sigma(D)r^{D-1}\,\mathrm{d}r = \int_{-R_0}^{R_0} \frac{\sigma(D)}{2}\,|\xi|^{D-1}\,\mathrm{d}\xi = \frac{\pi^{D/2}R_0^D}{\Gamma(1+D/2)}. \tag{2}$$

This agrees precisely with the spherical volume element for multi-dimensional Euclidean spaces [27] when D is a positive integer.

If we assume the system of units in such way that $\hbar = 1$, the one-dimensional momentum operator is

$$P = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\xi}. \tag{3}$$

Taking into account the integration weight (equation (1)), one can straightforwardly demonstrate that the momentum operator given by equation (3) is not Hermitian for $D \neq 1$. Therefore we must reject equation (3) for systems of a single degree of freedom in a fractional-dimensional space. Moreover, we have to assume more general commutation relations for the canonical variables than the well known relation

$$[\xi, P] = \mathrm{i} \tag{4}$$

since equation (4) leads, inevitably, to equation (3).

The most general wave-mechanical representation of the momentum operator [23, 24] can be found by considering the general Wigner commutation relations for the canonical variables of a Bose-like oscillator of one degree of freedom [22]:

$$\mathrm{i}P = [\xi, (P^2+\xi^2)/2] \qquad -\mathrm{i}\xi = [P, (P^2+\xi^2)/2]. \tag{5}$$

The relations above can be rewritten in the form [23, 24]

$$\{P, S\} = 0 \qquad \{\xi, S\} = 0 \tag{6}$$

where S is an operator given by

$$S = [\xi, P] - \mathrm{i}. \tag{7}$$

The eigenvalues ξ' of the coordinate operator ξ have a continuous spectrum ($-\infty < \xi' < \infty$) and the Hilbert space can be expanded in the eigenfunctions $|\xi'\rangle$ of ξ [24].

By sandwiching the S-operator with $|\xi'\rangle$ and $|\xi''\rangle$ the following expression can be obtained:

$$\langle\xi'|S|\xi''\rangle = (\xi' - \xi'')\langle\xi'|P|\xi''\rangle - \mathrm{i}\delta(\xi' - \xi''). \tag{8}$$

On the other hand, from the second relation in equation (6) the following expression results:

$$\langle\xi'|S|\xi''\rangle = 2\mathrm{i}A(\xi')\delta(\xi' + \xi'') \tag{9}$$

where $A(\xi')$ is in general a complex function.

The equation above together with equation (8) leads to

$$\langle\xi'|P|\xi''\rangle = -\mathrm{i}\delta'(\xi' - \xi'') + \mathrm{i}\frac{A(\xi')}{\xi'}\delta(\xi' + \xi'') + B(\xi')\delta(\xi' - \xi'') \tag{10}$$

with $B(\xi')$ a complex function.

Introducing, for convenience, the completeness condition $\int \mathrm{d}\xi'|\xi'\rangle\langle\xi'| = 1$ in equation (10) and writing the wavefunction for a state $|\cdots\rangle$ as $\Psi(\xi') = \langle\xi'|\cdots\rangle$, it can be found that

$$P\Psi(\xi') = -\mathrm{i}\frac{\mathrm{d}\Psi(\xi')}{\mathrm{d}\xi'} + \mathrm{i}\frac{A(\xi')}{\xi'}\Psi(-\xi') + B(\xi')\Psi(\xi'). \tag{11}$$

Thus the most general wave-mechanical representation of the momentum operator is then given by

$$P = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\xi} + \mathrm{i}\frac{A(\xi)}{\xi}R + B(\xi) \tag{12}$$

where R is the reflection operator.

In terms of R and taking into account equation (9), the S-operator may be rewritten as

$$S = 2\mathrm{i}A(\xi)R. \tag{13}$$

From the anti-hermiticity of S ($S = -S^{\dagger}$) and substituting equations (12) and (13) in the first relation of equation (6), the following restrictions on the undetermined functions $A(\xi)$ and $B(\xi)$ are found:

$$A^*(\xi) = A(-\xi) \tag{14}$$

and

$$\frac{\mathrm{d}A(\xi)}{\mathrm{d}\xi} + \mathrm{i}[B(\xi) + B(-\xi)]A(\xi) = 0. \tag{15}$$

With the aim to determine an expression for the fractional-dimensional momentum operator corresponding to a system of one degree of freedom, we now require the hermiticity of P [26]. Thus bearing in mind equation (12) and the integration weight (equation (1)), it is not difficult to find from the hermiticity of P that the function $B(\xi)$ must be given by

$$B(\xi) = -\mathrm{i}\frac{(D-1)}{2\xi}. \tag{16}$$

Equation (16) together with equation (15) leads to $A(\xi) = \text{constant}$. The value of this constant may be found if we bear in mind that $P = -\mathrm{i}\nabla$. Thus the nabla operator can be written as

$$\nabla = \frac{\mathrm{d}}{\mathrm{d}\xi} - \frac{A}{\xi}R + \frac{(D-1)}{2\xi}. \tag{17}$$

We then require $\nabla u = 0$ when $u = \text{constant}$ (and $u = \text{constant}$ if $\nabla u = 0$ (see appendix A)), and obtain

$$A = \frac{(D-1)}{2}. \tag{18}$$

The fractional-dimensional momentum operator for a system of a single degree of freedom is finally given by [26]

$$P = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\xi} + \mathrm{i}\frac{(D-1)}{2\xi}R - \mathrm{i}\frac{(D-1)}{2\xi}. \tag{19}$$

The operator above is, essentially, a particular case of the Dunkle operator [28]. However, one should notice that, as a consequence of the inclusion of the integration weight (equation (1)), our fractional-dimensional momentum operator is not a particular case of the momentum operator studied in [24, 25] and the differences may be important for defining a deformed calculus.

3. The momentum–position uncertainty relation

In a one-dimensional space, the Heisenberg uncertainty principle may be written as $\sqrt{\langle(\Delta\xi)^2\rangle\langle(\Delta p)^2\rangle} \geqslant 1/2$. This inequality is a consequence of the momentum-coordinate commutation relation (4). However, in a fractional-dimensional space equation (4) is no longer fulfilled, therefore we have to search for a more general momentum–position uncertainty relation.

We start with the introduction of an auxiliary positive integral depending on a real parameter α:

$$I(\alpha) = \frac{\sigma(D)}{2}\lim_{\gamma\to 0}\int_{-\infty}^{\infty} \mathrm{e}^{-\gamma\xi^2}\,|(\alpha\Delta\xi - \mathrm{i}\Delta p)\Psi|^2\,|\xi|^{D-1}\,\mathrm{d}\xi \geqslant 0 \tag{20}$$

where

$$\Delta\xi = \xi - \langle\xi\rangle \qquad \Delta p = p - \langle P\rangle \tag{21}$$

and $\langle\xi\rangle$, $\langle P\rangle$ are the expectation values of the position and momentum, respectively. It is worth remarking that in a fractional-dimensional space, the expectation value of a magnitude represented by an operator K for a given state described by the wavefunction Ψ is given by (see appendix B)

$$\langle K\rangle = \frac{\sigma(D)}{2}\lim_{\gamma\to 0}\int_{-\infty}^{\infty} \mathrm{e}^{-\gamma\xi^2}\Psi^* K\Psi|\xi|^{D-1}\,\mathrm{d}\xi. \tag{22}$$

From the expression above and making use of the hermiticity of the operators $\Delta\xi$ and Δp, equation (20) can be rewritten as

$$\begin{aligned} I(\alpha) &= \frac{\sigma(D)}{2}\lim_{\gamma\to 0}\int_{-\infty}^{\infty} \mathrm{e}^{-\gamma\xi^2}\Psi^*(\alpha\Delta\xi + \mathrm{i}\Delta p)(\alpha\Delta\xi - \mathrm{i}\Delta p)\Psi|\xi|^{D-1}\,\mathrm{d}\xi \\ &= \alpha^2\langle(\Delta\xi)^2\rangle - \mathrm{i}\alpha\langle[\xi, P]\rangle + \langle(\Delta p)^2\rangle \geqslant 0 \end{aligned} \tag{23}$$

where we have taken account of $[\Delta\xi, \Delta p] = [\xi, P]$, a relation that can be straightforwardly obtained from equation (21).

With the aim of guaranteeing the validity of the inequality (equation (23)) for any value of α we then require

$$\langle(\Delta\xi)^2\rangle\,\langle(\Delta p)^2\rangle \geqslant -\frac{\langle[\xi, P]\rangle^2}{4}. \tag{24}$$

After computing the expectation value $\langle[\xi, P]\rangle$ for even and odd states it is not difficult to prove that

$$\langle[\xi, P]\rangle = \begin{cases} \mathrm{i}D & \text{for even states} \\ \mathrm{i}(2-D) & \text{for odd states.} \end{cases} \tag{25}$$

The case of states without definite parity may be studied by introducing the normalized even $\Psi^{\text{even}}(\xi) = \sqrt{2}[\Psi(\xi)+\Psi(-\xi)]/2$ and odd $\Psi^{\text{odd}}(\xi) = \sqrt{2}[\Psi(\xi)-\Psi(-\xi)]/2$ functions. Thus bearing in mind that because of the symmetry properties, the even–odd mixed terms vanish (i.e. $\langle\Psi^{\text{even}}|\Psi^{\text{odd}}\rangle = \langle\Psi^{\text{odd}}|\Psi^{\text{even}}\rangle = 0$) and considering equation (25), the expectation value $\langle[\xi, P]\rangle$ for the state described by the wavefunction $\Psi(\xi) = \sqrt{2}[\Psi^{\text{even}}(\xi) + \Psi^{\text{odd}}(\xi)]/2$ is found to be

$$\langle[\xi, P]\rangle = \mathrm{i}. \tag{26}$$

From equations (24)–(26), the momentum–position uncertainty relations in a fractional-dimensional space are finally found:

$$\sqrt{\langle(\Delta\xi)^2\rangle}\sqrt{\langle(\Delta p)^2\rangle} \geqslant \begin{cases} D/2 & \text{for even states} \\ (2-D)/2 & \text{for odd states} \\ 1/2 & \text{otherwise.} \end{cases} \tag{27}$$

One should notice that for $D < 1$ the lower bound of the momentum–position uncertainty relation corresponding to an odd state is greater than that corresponding to an even state. An opposite behaviour takes place for $D > 1$. Thus even and odd states have similar behaviour only in the case $D = 1$. It is also remarkable that *classical* bounds corresponding to even and odd states are obtained when $D \to 0$ and $D \to 2$, respectively.

4. Free particle

Let us concentrate on the study of the behaviour of a free particle in a fractional-dimensional space. Once we have determined the momentum operator (see section 2), we can write the Hamiltonian operator as follows:

$$H = \frac{P^2}{2} = -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{(D-1)}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi} - \frac{(D-1)(1-R)}{2\xi^2}\right] \tag{28}$$

where we have considered units such that $\hbar = 1$, $m = 1$. The wavefunctions that describe the motion of a free particle can be found from the Schrödinger equation $H\Phi = E\Phi$. The eigenfunctions of the Hamiltonian operator may be expressed in terms of the even $\Phi^{\text{even}}(\xi) = \Phi(\xi) + \Phi(-\xi)$ and odd $\Phi^{\text{odd}}(\xi) = \Phi(\xi) - \Phi(-\xi)$ functions. They satisfy the following equations:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{(D-1)}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi} + 2E\right]\Phi^{\text{even}}(\xi) = 0 \tag{29}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{(D-1)}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi} - \frac{(D-1)}{\xi^2} + 2E\right]\Phi^{\text{odd}}(\xi) = 0 \tag{30}$$

respectively.

After solving equations (29), (30) and requiring $\Phi(\xi) = \left[\Phi^{\text{even}}(\xi) + \Phi^{\text{odd}}(\xi)\right]/2$, the wavefunctions are found to be

$$\Phi_p(\xi) = A_p\,|p\xi|^{1-D/2}\left[J_{D/2-1}(|p\xi|) + \mathrm{i}\,\mathrm{sgn}(p\xi)J_{D/2}(|p\xi|)\right] \qquad p = \sqrt{2E} \tag{31}$$

where

$$A_p = \sqrt{\frac{|p|^{D-1}}{2\sigma(D)}} \tag{32}$$

is a normalization factor (see appendix B),

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0 \end{cases} \tag{33}$$

and $J_\nu(x)$ are Bessel functions.

Bearing in mind that [29]

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x \tag{34}$$

one should note that the well known one-dimensional wavefunction $\Phi_p(\xi) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}p\xi}$ is immediately recovered when the value $D = 1$ is taken in equation (31).

An interesting picture of the behaviour of a free particle in a fractional-dimensional space may be now obtained by computing the position dependence of the probability density:

$$\rho_{\mathrm{p}} = \frac{\sigma(D)}{2}|\xi|^{D-1}\left|\Phi_p\right|^2 \tag{35}$$

for different values of the dimensional parameter. The results are displayed in figure 1. An oscillating behaviour of the probability density for $D \neq 1$ can be appreciated in spite of the well known constant value of ρ_{p} for the one-dimensional case. One should note that because of the inclusion of the integration weight (equation (1)) although $|\Phi_p|^2$ remains finite at $p\xi = 0$, $\rho_{\mathrm{p}}(0) \to \infty$ when $D < 1$. This situation leads to a very strong localization of the probability density at the origin (see figure 1(*a*)). However, when $D > 1$, the probability density is almost zero in the central region, so that a spreading of ρ_{p} may be clearly seen. This interesting behaviour reveals that the physical meaning of the dimensional parameter is strongly related to the degree of *compression* or *spread* of the probability density function.

5. The Bose-like oscillator

For the study of a fractional-dimensional Bose-like oscillator, we substitute equation (19) in $H = (P^2 + \xi^2)/2$ and obtain the corresponding expression for the Hamiltonian operator:

$$H = -\frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{(D-1)}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi} - \frac{(D-1)-(D-1)R}{2\xi^2} - \xi^2\right]. \tag{36}$$

Recently, certain classes of integrable many-body systems (the so-called Calogero models) have been analysed by applying the exchange operator formalism [30–33]. In this formalism, the Hamiltonian of the Calogero model is expressed in terms of the Dunkle operator and reduces, essentially, to the Hamiltonian given in (equation (36)). The similarity is remarkable and suggests a connection between the N-body Calogero problem and the fractional-dimensional Bose-like oscillator. Nevertheless, there is an essential physical difference. In the first case the presence of the Dunkle operator refers to a many-body problem and is a consequence of the interaction between the particles of the system, in the second case, however, the Dunkle operator appears as a consequence of the nature of the space.

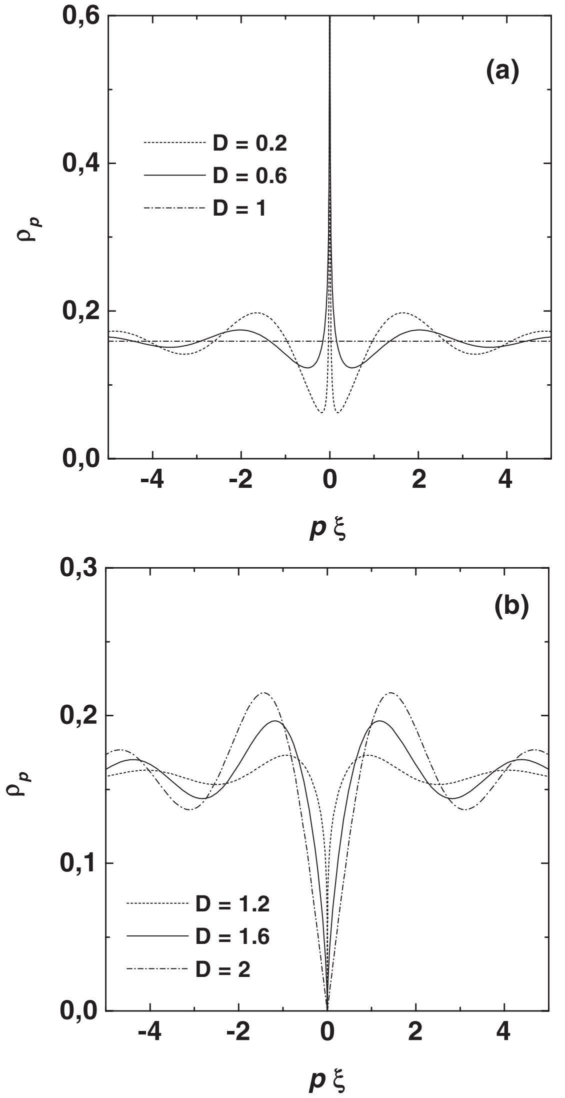

Figure 1. Position dependence of the probability density ρ_{p} corresponding to a free particle of one degree of freedom in a fractional-dimensional space for different values of the dimensional parameter. (*a*) For $D \leqslant 1$ and (*b*) for $D > 1$.

Before discussing the eigenvalue problem of the Hamiltonian it is convenient to note that

$$[R, H] = 0. \tag{37}$$

The parity of a state is thus an integral of motion, so that the eigenvalue problem of the Hamiltonian can be expressed by

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{(D-1)}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi} - \xi^2\right]\chi^{\mathrm{even}}(\xi) = -2E\chi^{\mathrm{even}}(\xi) \tag{38}$$

and

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{(D-1)}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi} - \frac{(D-1)}{\xi^2} - \xi^2\right]\chi^{\mathrm{odd}}(\xi) = -2E\chi^{\mathrm{odd}}(\xi) \tag{39}$$

for even and odd states respectively.

The wavefunctions of the fractional-dimensional Bose-like oscillator of a single degree of freedom are then given by

$$\chi_n(\xi) = \sqrt{\frac{2}{\sigma(D)}}\phi_n^{\frac{D-1}{2}}(\xi) = \begin{cases} A_n^{\mathrm{even}}(D)\exp[-\xi^2/2]\, L_{\frac{n}{2}}^{D/2-1}(\xi^2) & \text{for even states} \\ A_n^{\mathrm{odd}}(D)\exp[-\xi^2/2]\,\xi\, L_{\frac{n-1}{2}}^{D/2}(\xi^2) & \text{for odd states} \end{cases} \tag{40}$$

where $\phi_n^{\frac{D-1}{2}}(x)$ are the generalized Hermite functions (for an extensive discussion of the definition and properties of the generalized Hermite functions and their relations to the generalized Hermite and generalized Laguerre polynomials you can see [34]), L_k^α are the generalized Laguerre polynomials, and

$$A_n^{\mathrm{even}}(D) = (-1)^{n/2}\left[\frac{(n/2)!\Gamma(D/2)}{\pi^{D/2}\Gamma(\frac{n+D}{2})}\right]^{1/2} \tag{41}$$

$$A_n^{\mathrm{odd}}(D) = (-1)^{\frac{n-1}{2}}\left[\frac{(\frac{n-1}{2})!\Gamma(D/2)}{\pi^{D/2}\Gamma(\frac{n+D+1}{2})}\right]^{1/2}. \tag{42}$$

The wavefunctions satisfy the normalization condition

$$\frac{\sigma(D)}{2}\int_{-\infty}^{\infty}|\chi(\xi)|^2|\xi|^{D-1}\,\mathrm{d}\xi = 1. \tag{43}$$

The corresponding eigenenergies are given by

$$E_n = n + D/2 \qquad (n = 0, 1, 2, 3, \ldots) \tag{44}$$

an expression that is in agreement with a previous work of Stillinger [11].

Bearing in mind that the generalized Hermite functions $\phi_n^0(x)$ reduce to Hermite functions [34], we must note that equation (40) recovers the well known one-dimensional case when $D = 1$.

The probability of finding the value of the *pseudocoordinate* of a particle (in the nth state) within the interval $(\xi, \xi + \mathrm{d}\xi)$ is proportional to the probability density

$$\rho_n(\xi) = \frac{\sigma(D)}{2}|\xi|^{D-1}|\chi_n(\xi)|^2 \qquad (n = 0, 1, 2, 3, \ldots). \tag{45}$$

In figure 2 we present the probability density ρ_0, corresponding to the ground state of a fractional-dimensional Bose-like oscillator as a function of the *pseudocoordinate* ξ for different values of the dimensionality. When $D < 1$ (cf figure 2(a)), one may notice that the *compression* of the probability density ρ_0 around the origin of *pseudocoordinates* increases as the dimension decreases. Moreover, when $D > 1$ (cf figure 2(b)), ρ_0 becomes vanishingly small in the central region, so that a *spreading* of the probability density is quite apparent (especially when the dimensionality increases). This trend—similar to that of the free particle (see the section above)—suggests that an one-dimensional harmonic oscillator in presence of a confining (spreading) potential could be treated as a fractional-dimensional Bose-like oscillator of one degree of freedom with dimension $D < 1$ ($D > 1$) essentially related to the degree of compression (spread) of the probability density. The above statements are in agreement with the approach introduced by He [12] and extensively used in low-dimensional condensed matter physics [12–21].

The strong localization of the particle around $\xi = 0$ is notable when $D < 1$ (see figure 2(a)). In fact, the probability density ρ_0 becomes infinite at $\xi = 0$ when $D < 1$. This is because the integration weight (equation (1)) diverges at the origin of *pseudocoordinates* when $D < 1$ whereas the wavefunction remains finite.

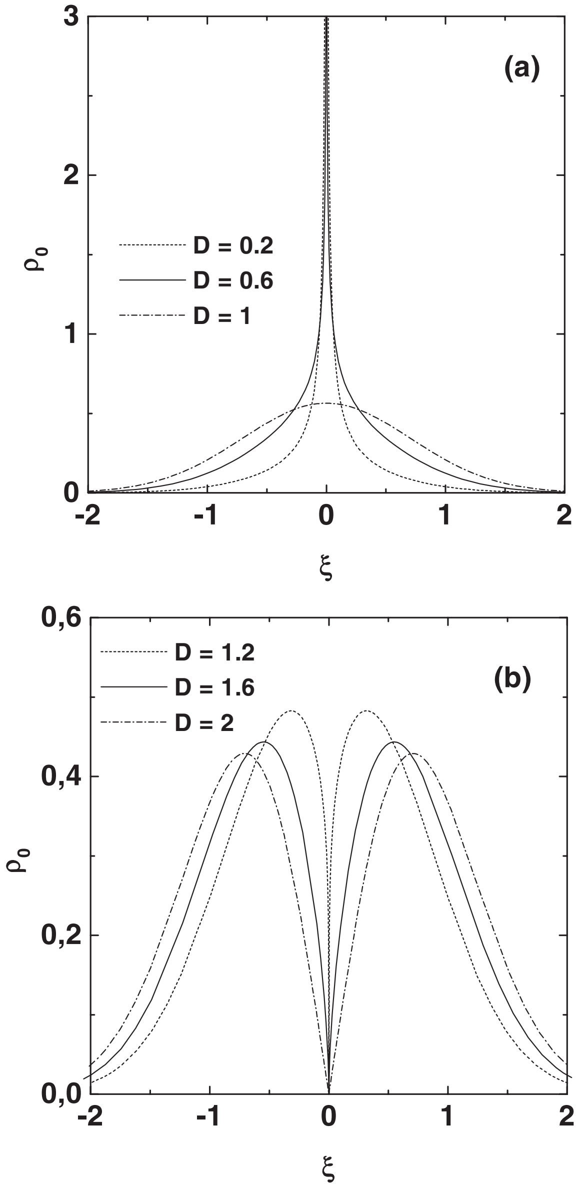

Figure 2. The probability density ρ_0 corresponding to the ground state of a fractional-dimensional Bose-like oscillator as a function of the *pseudocoordinate* and for different values of the dimensional parameter. (*a*) For $D \leqslant 1$ and (*b*) for $D > 1$.

The *pseudocoordinate* dependence of the probability density ρ_1 corresponding to the first excited state is displayed in figure 3 for different values of the dimensional parameter. Because of the odd parity of the state the probability density becomes zero at the origin of *pseudocoordinate* and there is no longer localization in the central region. However, as in the cases above, the existence of compression (spread) of ρ_1 for $D < 1$ ($D > 1$) is quite apparent.

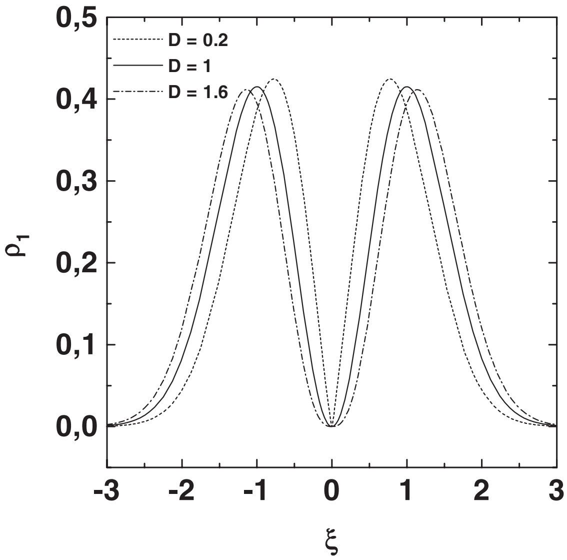

Figure 3. Same as in figure 2, for the first excited state ($n = 1$).

6. Bose-like operators

The annihilation and creation fractional-dimensional Bose-like operators are given by

$$a = \frac{1}{\sqrt{2}}(\xi + \mathrm{i}P) = \frac{1}{\sqrt{2}}\left[\frac{\mathrm{d}}{\mathrm{d}\xi} + \xi - \frac{(D-1)}{2\xi}R + \frac{(D-1)}{2\xi}\right] \tag{46}$$

and

$$a^{\dagger} = \frac{1}{\sqrt{2}}(\xi - \mathrm{i}P) = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}\xi} + \xi + \frac{(D-1)}{2\xi}R - \frac{(D-1)}{2\xi}\right] \tag{47}$$

respectively. From equations above and using the following relations for generalized Laguerre polynomials: (see [35])

$$\frac{\mathrm{d}L_k^{\alpha}(x)}{\mathrm{d}x} = -L_{k-1}^{\alpha+1}(x) = \frac{1}{x}\left[(k+\alpha)L_k^{\alpha-1}(x) - \alpha L_k^{\alpha}(x)\right] \tag{48}$$

and

$$\frac{\mathrm{d}L_k^{\alpha}(x)}{\mathrm{d}x} = L_k^{\alpha}(x) - L_k^{\alpha+1}(x) = \frac{1}{x}\left[(k+1)L_{k+1}^{\alpha-1}(x) - (\alpha - x)L_k^{\alpha}(x)\right] \tag{49}$$

it is not difficult to prove that

$$a\chi_n = \begin{cases} \sqrt{n}\ \chi_{n-1} & \text{for} \quad n \text{ even} \\ \sqrt{n+D-1}\ \chi_{n-1} & \text{for} \quad n \text{ odd} \end{cases} \tag{50}$$

and

$$a^{\dagger}\chi_n = \begin{cases} \sqrt{n+D}\ \chi_{n+1} & \text{for} \quad n \text{ even} \\ \sqrt{n+1}\ \chi_{n+1} & \text{for} \quad n \text{ odd.} \end{cases} \tag{51}$$

It is a remarkable result that, in contrast with the *a priori* assumption that the annihilation and creation operators are the same in fractional- and one-dimensional spaces (see [20]), equations (50) and (51) show us a clear dependence of the Bose-like operators on the dimensional parameter.

Moreover, from equations (46), (47) and bearing in mind the orthonormality relation (equation (43)) one may obtain

$$\langle \xi^2 \rangle_n = n + D/2. \tag{52}$$

One should observe that if $D = 0$, both the energy and the mean square amplitude of the zero-point oscillations may reach—as in the classical linear oscillator—the value zero. This is because the integration weight in fractional-dimensional space becomes a Dirac delta function $\delta(\xi)$ [11] when $D \to 0$ i.e. it collapses to zero extension, whereas χ_0 remains bounded. Consequently, the probability density $\rho_0 \to \delta(\xi)$ when $D \to 0$ and the particle is totally localized at $\xi = 0$.

In terms of the fractional-dimensional Bose-like operators, the Hamiltonian may be written as usual:

$$H = \frac{a^\dagger a + a a^\dagger}{2}. \tag{53}$$

However, the standard definition for the operator number of particles has to be generalized:

$$N = \tfrac{1}{2}(\{a, a^\dagger\} - D). \tag{54}$$

The fractional-dimensional creation–annihilation operators $a^\dagger$, a are also found to satisfy

$$[a, a^\dagger] = 1 + (D - 1)R \qquad \{R, a\} = \{R, a^\dagger\} = 0 \tag{55}$$

and the trilinear parabosonic commutation relations

$$[\{a, a^\dagger\}, a] = -2a \qquad [\{a, a^\dagger\}, a^\dagger] = 2a^\dagger. \tag{56}$$

In equation (55) the quantity $D - 1$ can be understood as a real deformation parameter of a deformed Heisenberg algebra with reflection [36, 37]. Thus, at the integer values of the dimensionality $D = 1, 2, 3, \ldots$, the algebra (55) represents parabosons of order D [38, 39]. On the other hand, for non-integer values of the dimensionality ($D > 0$), the algebra (55) can be considered as a generalization of parabosons.

For spaces with $0 < D < 2$, the commutation relations (55) can be normalized (in a similar way as in [37]) by introducing the new operators

$$c = a[1 - (D - 1)R]^{-1/2} \qquad c^\dagger = [1 - (D - 1)R]^{-1/2} a^\dagger. \tag{57}$$

These operators anticommute with the reflection operator, $\{R, c\} = \{R, c^\dagger\} = 0$, and satisfy the relation

$$cc^\dagger - (2 - D)^R D^{-R} c^\dagger c = 1 \qquad 0 < D < 2. \tag{58}$$

One should note that the normalized form (equation (58)) represents a guon-like algebra [40,41] with a D-dependent operator instead of a c-number q-factor.

7. Conclusions

Taking account of the general Wigner commutation relations, we have obtained momentum–position uncertainty relations for systems of one degree of freedom in a fractional-dimensional space. It is shown that the lower bound corresponding to the momentum–position uncertainty inequality depends on the dimensional parameter for even and odd states. However, for states with nondefinite parity, the well known momentum–position uncertainty relation of the one-dimensional case is recovered independently of the dimension of the space. The behaviour of the probability density in fractional-dimensional space has been studied for two systems: a free particle and a Bose-like oscillator. In both cases, a compression (spread) of the probability density function can be observed for $D < 1$ ($D > 1$). This interesting behaviour reveals a strong relation between the dimension of the space and the degree of confinement (or spread) of the system, a result in agreement with a previous hypothesis extensively used in modelling low-dimensional systems [12–21]. Finally, we have also deduced expressions for annihilation and creation Bose-like operators in a fractional-dimensional space. These operators together with the reflection operator form an R-deformed Heisenberg algebra with a deformation parameter depending on the dimension of the space. The normalized form of the corresponding R-deformed Heisenberg algebra represents a guon-like algebra with a D-dependent operator instead of a c-number q-factor.

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Appendix A

Here we solve the equation

$$\nabla u = \frac{\mathrm{d}u}{\mathrm{d}\xi} - \frac{A}{\xi} R u + \frac{(D-1)}{2\xi} u = 0 \tag{59}$$

and demonstrate that the general solution of equation above reduces to a constant if we chose $A = \frac{(D-1)}{2}$.

By introducing the functions $u^{\mathrm{even}}(\xi) = u(\xi) + u(-\xi)$ and $u^{\mathrm{odd}}(\xi) = u(\xi) - u(-\xi)$ and considering equation (59) the following equations can be found:

$$\left[\frac{\mathrm{d}}{\mathrm{d}\xi} - \frac{A}{\xi} + \frac{(D-1)}{2\xi}\right] u^{\mathrm{even}}(\xi) = 0 \tag{60}$$

and

$$\left[\frac{\mathrm{d}}{\mathrm{d}\xi} + \frac{A}{\xi} + \frac{(D-1)}{2\xi}\right] u^{\mathrm{odd}}(\xi) = 0. \tag{61}$$

After solving equations (60) and (61) we obtain general expressions for $u^{\mathrm{even}}(\xi)$ and $u^{\mathrm{odd}}(\xi)$

$$u^{\mathrm{even}}(\xi) = c_1 |\xi|^{\frac{2A-(D-1)}{2}} \qquad u^{\mathrm{odd}}(\xi) = c_2 |\xi|^{-\frac{2A-(D-1)}{2}} \qquad (c_1, c_2 = \mathrm{const}). \tag{62}$$

From the requirement $u^{\mathrm{odd}}(\xi) = -u^{\mathrm{odd}}(-\xi)$ the constant c_2 is found to be zero. Finally, the general solution $u = [u^{\mathrm{even}} + u^{\mathrm{odd}}]/2$ of equation (59) is obtained:

$$u(\xi) = \frac{c_1}{2} |\xi|^{\frac{2A-(D-1)}{2}}. \tag{63}$$

If we now put $A = \frac{(D-1)}{2}$ in equation (63) the general solution of equation (59) reduces to $u = \mathrm{constant}$.

Appendix B

One should notice that the eigenfunctions $\Phi_p(\xi)$ of the fractional-dimensional momentum operator (see equation (31)) are not square integrable. Then by introducing a damping factor in the inner products as in [24]

$$\langle \Phi_p | \Phi_{p'} \rangle = \frac{\sigma(D)}{2} \lim_{\gamma \to 0} \int_{-\infty}^{\infty} \mathrm{e}^{-\gamma \xi^2} \Phi_p^*(\xi) \Phi_{p'}(\xi) |\xi|^{D-1} \, \mathrm{d}\xi \qquad (\gamma > 0) \tag{64}$$

one can demonstrate that the eigenfunctions $\Phi_p(\xi)$ form a complete, orthonormalized system.

Making use of the formula (see [24])

$$\lim_{\gamma\to 0}\int_0^\infty \mathrm{d}\xi\, \mathrm{e}^{-\gamma\xi^2}\big[|p\xi|^{1/2}J_\alpha(|p\xi|)\big]\big[|p'\xi|^{1/2}J_\alpha(|p'\xi|)\big]$$
$$= \lim_{\gamma\to 0}\frac{|pp'|^{1/2}}{2\gamma}\exp\left[-\frac{|p|^2+|p'|^2}{4\gamma}\right]I_\alpha\left(\frac{|pp'|}{2\gamma}\right)$$
$$= \lim_{\gamma\to 0}\frac{1}{2\sqrt{\pi\gamma}}\exp\left[-\frac{(|p|-|p'|)^2}{4\gamma}\right] \tag{65}$$

where $I_\alpha(x)$ represents a modified Bessel function with the asymptotic behaviour $I_\alpha(x) \to \frac{\mathrm{e}^x}{\sqrt{2\pi x}}$ as $x \to \infty$ and noting that the last term in equation (65) is just a representation of the Dirac delta function $\delta(|p| - |p'|)$, it is not difficult to obtain from equations (31), (64) the following relation:

$$\langle\Phi_p|\Phi_{p'}\rangle = 2\sigma(D)|pp'|^{\frac{1-D}{2}}A_pA_{p'}\delta(p-p'). \tag{66}$$

By now choosing the normalization factor as

$$A_p = \sqrt{\frac{|p|^{D-1}}{2\sigma(D)}} \tag{67}$$

equation (66) reduces to the orthonormalization condition $\langle\Phi_p|\Phi_{p'}\rangle = \delta(p - p')$. Following a similar procedure, the completeness condition $\langle\Phi_p(\xi)|\Phi_p(\xi')\rangle = \delta(\xi - \xi')$ can be also obtained.

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